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COMMENT

Inverse relation for the two-dimensional Ising model with multispin interactions

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Abstract. We show that the partition function per site $\mathcal{K}_m(K_x, K_\tau)$ of the square lattice Ising model with m-spin interaction K_x and two-spin interaction K_τ satisfies the inverse relation $\mathcal{K}_m(K_x, K_\tau)\mathcal{K}_m(-K_x, K_\tau \pm i\pi/2) = \pm 2i \sinh 2K_\tau$ for all m values.

The two-dimensional Ising model with m-spin interactions (Turban 1982a, b, Penson *et al* 1982) with Hamiltonian

$$-\beta\mathcal{H}_m = K_x \sum_{i,j} \prod_{k=0}^{m-1} \sigma_{i-k,j} + K_\tau \sum_{i,j} \sigma_{i,j}\sigma_{i,j+1} \tag{1}$$

where $K_x(K_\tau)$ gives the interaction in the spatial (temporal) direction on the square lattice, has been extensively studied in the past few years in its 2D classical or 1D quantum mechanical version. The model is known to be self-dual:

$$\mathcal{K}_m(K_x, K_\tau) = (\sinh 2K_x \sinh 2K_\tau)^{1/2}\mathcal{K}_m(\tilde{K}_x, \tilde{K}_\tau) \tag{2}$$

where $\mathcal{K}_m$ is the partition function per site and the dual interactions $\tilde{K}_x$ and $\tilde{K}_\tau$ are the usual ones:

$$\sinh 2K_{x(\tau)} \sinh 2\tilde{K}_{\tau(x)} = 1. \tag{3}$$

It follows that the critical line is given by

$$\sinh 2K_{xc} \sinh 2K_{\tau c} = 1 \tag{4}$$

for all m. Symmetry and ground state degeneracy considerations led us to conjecture that the transition is of first order when $m > 3$ and belongs to the Potts $q = 4$ universality class when $m = 3$. This is supported by finite-size scaling studies (Penson *et al* 1982, Debierre and Turban 1983, Vanderzande and Iglói 1987), weak and strong coupling expansions (Iglói *et al* 1986), conformal invariance (Kolb and Penson 1986, Alcaraz and Barber 1987, Iglói 1987, Penson *et al* 1988), Monte Carlo simulations (Blöte *et al* 1986, Alcaraz 1987) and mapping of the $m = 3$ model in the extreme anisotropic limit on the $q = 4$ Potts model (Blöte 1987).

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In this comment we show that besides the duality relation the m-spin Ising model partition function also satisfies an inverse relation (Baxter 1980, Jaekel and Maillard 1981, Maillard 1983) in the temporal direction. Let us introduce an interaction-round-a-face (IRF) model built on two replicas of the m-spin Ising model with a relative shift in the diagonal direction as shown in figure 1. On the replicas the μ variables correspond to $m-1$ successive Ising spins σ grouped together along the spatial direction. To each face of the IRF model we associate the statistical weight

$$W(\mu_1, \mu_2, \mu_3, \mu_4) = \prod_{k=0}^{m-2} a^{\prod_{l=0}^{m-1} \sigma_{i_1+k+l, j_1}} \prod_{k=0}^{m-2} b^{\sigma_{i_2+k, i_2}\sigma_{i_2+k, i_2+1}} \tag{5a}$$

$$\mu = \{\sigma_{i,j}; \sigma_{i+1,j}; \ldots; \sigma_{i+m-2,j}\} \tag{5b}$$

$$a = \mathrm{e}^{K_x} \qquad b = \mathrm{e}^{K_\tau} \tag{5c}$$

where the $\{\mu\}$ are coupled in pairs through $m-1$ interactions in both directions. Consider now two adjacent IRF models with $N/2$ sites, statistical weights W and $\bar{W}$, with an interface in the spatial direction (figure 2). The new interactions $\bar{K}_x$ and $\bar{K}_\tau$ are such that the inverse relation

$$\sum_{\mu_2'} W(\mu_1, \mu_2', \mu_3, \mu_4)\bar{W}(\mu_1, \mu_2, \mu_3, \mu_2') = \Lambda\delta(\mu_2, \mu_4) \tag{6}$$

is satisfied for all the external spin configurations $\mu_1, \mu_2, \mu_3, \mu_4$. With $\bar{K}_x = -K_x$, one gets $a\bar{a} = 1$ so that μ_1 and μ_3 are eliminated. Then equation (6) is obtained if $\bar{K}_\tau$ is chosen to satisfy

$$\sum_{\sigma_2'} \bar{b}^{\sigma_2\sigma_2'} b^{\sigma_2'\sigma_4} = \lambda\delta(\sigma_2, \sigma_4) \tag{7}$$

with $\Lambda = \lambda^{m-1}$ since equation (6) contains $m-1$ factors of this type, one for each of the $m-1$ spins in the μ variables. Summing over σ_2' leads to

$$\bar{b}^{\sigma_2} b^{\sigma_4} + \bar{b}^{-\sigma_2} b^{-\sigma_4} = \lambda\delta(\sigma_2, \sigma_4) \tag{8}$$

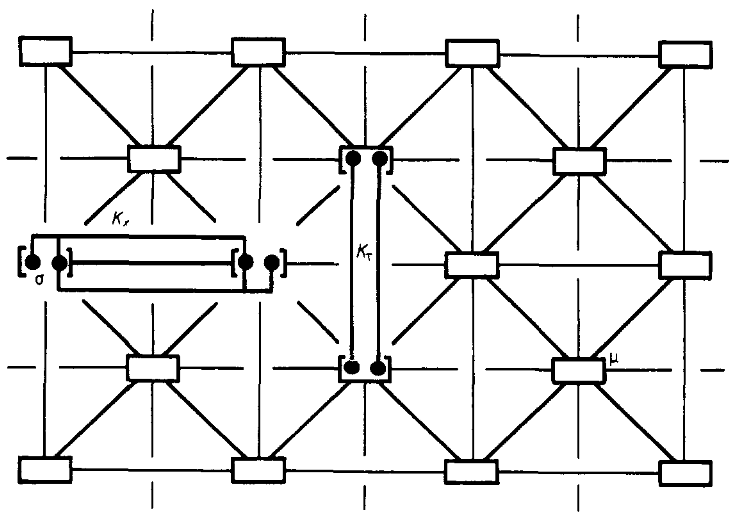

Figure 1. An IRF model (heavy lines) is constructed on two replicas of the m-spin Ising model (here with $m = 3$). The variables of the IRF model are groups of $m-1$ successive Ising spins σ in the spatial direction. They are coupled by $m-1$ interactions K_x and K_τ.

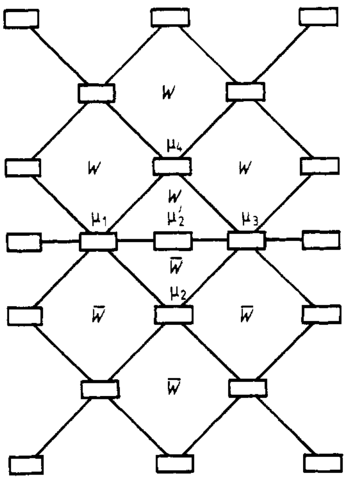

Figure 2. Two adjacent IRF models with statistical weights W and $\bar{W}$, the interface lying in the spatial direction. Through a repeated summation over the central states such as μ_2' on the interface, the product of the partition functions of the IRF models is obtained.

and

$$b\bar{b}+(b\bar{b})^{-1}=\lambda \qquad (\sigma_2=\sigma_4) \tag{9a}$$

$$b/\bar{b}+\bar{b}/b=0 \qquad (\sigma_2=-\sigma_4) \tag{9b}$$

so that

$$\lambda=\pm i(b^2-b^{-2})=\pm 2\mathrm{i}\sinh 2K_\tau \tag{10a}$$

$$\bar{b}=\pm ib \qquad \bar{K}_\tau=K_\tau\pm \mathrm{i}\pi/2. \tag{10b}$$

A repeated summation over the central states at the interface gives the product of the partition functions in the thermodynamic limit

$$\mathcal{K}_m(a,b)^{N(m-1)/2}\mathcal{K}_m(a^{-1},\pm ib)^{N(m-1)/2}=\Lambda^{N/2}=\lambda^{N(m-1)/2} \tag{11}$$

since there are $N(m-1)/2$ spins on each side and a factor Λ eliminates $2(m-1)$ spins. It follows that the partition function per site satisfies the inverse relation

$$\mathcal{K}_m(K_x,K_\tau)\mathcal{K}_m(-K_x,K_\tau\pm\mathrm{i}\pi/2)=\pm 2\mathrm{i}\sinh 2K_\tau \tag{12}$$

for all m.

The self-duality relation and the inverse relation keep the same form as for the square lattice Ising model ($m=2$) but in order to progress towards a solution of the m-spin model, m-dependent relations are needed. The model is anisotropic and one should look for an inverse relation working with the interface in the temporal direction. We hope that this comment will stimulate further work in this direction.

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